# 7
# NATURAL DEDUCTION IN PROPOSITIONAL LOGIC

## 7.1 RULES OF IMPLICATION I

**Natural deduction** is a method for establishing the validity of propositional type arguments that is both simpler and more enlightening than the method of truth tables. By means of this method, the conclusion of an argument is actually derived from the premises through a series of discrete steps. In this respect natural deduction resembles the method used in geometry to derive theorems relating to lines and figures; but whereas each step in a geometrical proof depends on some mathematical principle, each step in a logical proof depends on a **rule of inference.** Eighteen rules of inference will be set forth in this chapter. The first four should be familiar from the previous chapter:

1. *Modus ponens* (MP):

$$\begin{array}{l} p \supset q \\ \underline{p} \\ q \end{array}$$

2. *Modus tollens* (MT):

$$\begin{array}{l} p \supset q \\ \underline{\sim q} \\ \sim p \end{array}$$

3. Hypothetical syllogism (HS):

$$\begin{array}{l} p \supset q \\ \underline{q \supset r} \\ p \supset r \end{array}$$

4. Disjunctive syllogism (DS):

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ q \end{array}$$

For an illustration of the use of three of these rules, consider the following argument:

> If the Astros win the playoff, then the Braves will lose the pennant. If the Astros do not win the playoff, then either Connolly or Davis will be fired. The Braves will not lose the pennant. Furthermore, Connolly will not be fired. Therefore, Davis will be fired.

The first step is to symbolize the argument, numbering the premises and writing the conclusion to the right of the last premise, separated by a slash mark:

1. $A \supset B$
2. $\sim A \supset (C \vee D)$
3. $\sim B$
4. $\sim C$    / $D$

The conclusion is now derived from the premises via steps 5 through 7. The justification for each line is written to the immediate right:

| | | |
|---|---|---|
| 5. | $\sim A$ | 1, 3, MT |
| 6. | $C \vee D$ | 2, 5, MP |
| 7. | $D$ | 4, 6, DS |

Line 5 is obtained from lines 1 and 3 via *modus tollens.* In other words, when $A$ and $B$ in these lines are substituted respectively for the $p$ and $q$ of the *modus tollens* rule, line 5 follows as the conclusion. Then, when $\sim A$ and $C \vee D$ in lines 2 and 5 are substituted respectively for the $p$ and $q$ of the *modus ponens* rule, line 6 follows as the conclusion. Finally, when $C$ and $D$ in lines 4 and 6 are substituted respectively for the $p$ and $q$ of the disjunctive syllogism rule, line 7 follows as the final conclusion. These lines constitute a valid derivation of the conclusion from the premises because each line is a substitution instance of a valid argument form.

These arguments are all instances of ***modus ponens*** (MP)

| | | |
|---|---|---|
| $\sim F \supset (G \equiv H)$ | $(A \vee B) \supset \sim (C \cdot D)$ | $K \cdot L$ |
| $\underline{\sim F}$ | $\underline{A \vee B}$ | $\underline{(K \cdot L) \supset [(R \supset S) \cdot (T \supset U)]}$ |
| $G \equiv H$ | $\sim (C \cdot D)$ | $(R \supset S) \cdot (T \supset U)$ |

Here is an example of another completed proof. The conclusion to be obtained is written to the right of the last premise (line 4). Lines 5 through 7 are used to derive the conclusion:

1. $F \supset G$
2. $F \vee H$
3. $\sim G$
4. $H \supset (G \supset I)$     / $F \supset I$
5. $\sim F$     1, 3, MT
6. $H$     2, 5, DS
7. $G \supset I$     4, 6, MP
8. $F \supset I$     1, 7, HS

When the letters in lines 1 and 3 are substituted into the *modus tollens* rule, line 5 is obtained. Then, when the letters in lines 2 and 5 are substituted into the disjunctive syllogism rule, line 6 is obtained. Line 7 is obtained by substituting $H$ and $G \supset I$ from lines 4 and 6 into the *modus ponens* rule. Finally, line 8 is obtained by substituting the letters in lines 1 and 7 into the hypothetical syllogism rule. Notice that the conclusion, stated to the right of line 4, is not (and never is) part of the proof. It merely indicates what the proof is supposed to yield in the end.

These arguments are all instances of ***modus tollens*** (MT)

| | | |
|---|---|---|
| $(D \vee F) \supset K$ | $\sim G \supset \sim (M \vee N)$ | $\sim T$ |
| $\sim K$ | $\sim\sim (M \vee N)$ | $[(H \vee K) \cdot (L \vee N)] \supset T$ |
| $\sim (D \vee F)$ | $\sim\sim G$ | $\sim [(H \vee K) \cdot (L \vee N)]$ |

The successful use of natural deduction to derive a conclusion from one or more premises depends on the ability of the reasoner to visualize more or less complex arrangements of atomic propositions as instances of the basic rules of inference. Here is a slightly more complex example:

1. $\sim (A \cdot B) \vee [\sim (E \cdot F) \supset (C \supset D)]$
2. $\sim\sim (A \cdot B)$
3. $\sim (E \cdot F)$
4. $D \supset G$     / $C \supset G$
5. $\sim (E \cdot F) \supset (C \supset D)$     1, 2, DS
6. $C \supset D$     3, 5, MP
7. $C \supset G$     4, 6, HS

Line 4 is the last premise. To obtain line 5, $\sim (A \cdot B)$ and $[\sim (E \cdot F) \supset (C \supset D)]$ are substituted respectively for the $p$ and $q$ of the disjunctive syllogism rule, yielding $[\sim (E \cdot F) \supset (C \supset D)]$ as the conclusion. Next, $\sim (E \cdot F)$ and $C \supset D$ are substituted respectively for the $p$ and $q$ of *modus ponens*, yielding $C \supset D$ on line 6. Finally, lines 6 and 4 are combined to yield line 7 via the hypothetical syllogism rule.

The proofs that we have investigated thus far have been presented in ready-made form. We turn now to the question of how the various lines are obtained, leading in the end to the conclusion. What strategy is used in deriving these lines? While the answer is somewhat complex, there

These arguments are all instances of pure **hypothetical syllogism** (HS)

| | | |
|---|---|---|
| $A \supset (D \cdot F)$ | $\sim M \supset (R \supset S)$ | $(L \supset N) \supset [(S \vee T) \cdot K]$ |
| $(D \cdot F) \supset \sim H$ | $(C \vee K) \supset \sim M$ | $(C \equiv F) \supset (L \supset N)$ |
| $A \supset \sim H$ | $(C \vee K) \supset (R \supset S)$ | $(C \equiv F) \supset [(S \vee T) \cdot K]$ |

are a few basic rules of thumb that should be followed. Always begin by looking at the conclusion and by then attempting to locate the conclusion in the premises. Let us suppose that the conclusion is a single letter $L$. We begin by looking for $L$ in the premises. Let us suppose we find it in a premise that reads:

$$K \supset L$$

Immediately we see that we can obtain $L$ via *modus ponens* if we first obtain $K$. We now begin searching for $K$. Let us suppose that we find $K$ in another premise that reads

$$J \vee K$$

From this we see that we could obtain $K$ via disjunctive syllogism if we first obtain $\sim J$. The process continues until we isolate the required statement on a line by itself. Let us suppose that we find $\sim J$ on a line by itself. The thought process is then complete, and the various steps may be written out in the reverse order in which they were obtained mentally. The proof would look like this:

| | |
|---|---|
| 1. $\sim J$ | |
| 2. $J \vee K$ | |
| 3. $K \supset L$ | / $L$ |
| 4. $K$ | 1, 2, DS |
| 5. $L$ | 3, 4, MP |

These arguments are all instances of **disjunctive syllogism** (DS)

| | | |
|---|---|---|
| $U \vee \sim(W \cdot X)$ | $\sim(E \vee F)$ | $\sim B \vee [(H \supset M) \cdot (S \supset T)]$ |
| $\sim U$ | $(E \vee F) \vee (N \supset K)$ | $\sim\sim B$ |
| $\sim(W \cdot X)$ | $N \supset K$ | $(H \supset M) \cdot (S \supset T)$ |

Turning now to a different example, let us suppose that the conclusion is the conditional statement $R \supset U$. We begin by attempting to locate $R \supset U$ in the premises. If we cannot find it, we look for its separate components, $R$ and $U$. Let us suppose we find $R$ in the antecedent of the conditional statement

$$R \supset S$$

Furthermore, let us suppose we find $U$ in the consequent of the conditional statement

$$T \supset U$$

We then see that we can obtain $R \supset U$ via a series of hypothetical syllogism steps if we first obtain $S \supset T$. Let us suppose that we find $S \supset T$ on a line by itself. The proof has now been completely thought through and may be written out as follows:

| | |
|---|---|
| 1. $S \supset T$ | |
| 2. $T \supset U$ | |
| 3. $R \supset S$ | / $R \supset U$ |
| 4. $R \supset T$ | 1, 3, HS |
| 5. $R \supset U$ | 2, 4, HS |

At this point a word of caution is in order about the meaning of a proposition being "obtained." Let us suppose that we are searching for $E$ and we find it in a premise that reads $E \supset F$. The mere fact that we have located the letter $E$ in this line does not mean that we have obtained $E$. $E \supset F$ means that *if* we have $E$, then we have $F$; it does *not* mean that we *have* either $E$ or $F$. From such a line we could obtain $F$ (via *modus ponens*) if we first obtain $E$, or, we could obtain $\sim E$ (via *modus tollens*) if we first obtain $\sim F$. $E \supset F$ by itself gives us nothing, and even if we combine it with other lines, there is no way that we could ever obtain $E$ from such a line.

Here is a sample argument:

| | |
|---|---|
| 1. $A \vee B$ | |
| 2. $\sim C \supset \sim A$ | |
| 3. $C \supset D$ | |
| 4. $\sim D$ | / $B$ |

We begin by searching for $B$ in the premises. Finding it in line 1, we see that it can be obtained via disjunctive syllogism if we first obtain $\sim A$. This in turn can be gotten from line 2 via *modus ponens* if we first obtain $\sim C$, and this can be gotten from line 3 via *modus tollens* once $\sim D$ is obtained. Happily, the latter is stated by itself on line 4. The proof has now been completely thought through and can be written out as follows:

| | |
|---|---|
| 1. $A \vee B$ | |
| 2. $\sim C \supset \sim A$ | |
| 3. $C \supset D$ | |
| 4. $\sim D$ | / $B$ |
| 5. $\sim C$ | 3, 4, MT |

6. $\sim A$      2, 5, MP
7. $B$      1, 6, DS

Another example:

1. $E \supset (K \supset L)$
2. $F \supset (L \supset M)$
3. $G \vee E$
4. $\sim G$
5. $F$      / $K \supset M$

We begin by searching for $K \supset M$ in the premises. Not finding it, we search for the separate components, $K$ and $M$, and locate them in lines 1 and 2. The fact that $K$ appears in the antecedent of a conditional statement, and $M$ in the consequent of another, immediately suggests hypothetical syllogism. But first we must obtain these conditional statements on lines by themselves. We can obtain $K \supset L$ via *modus ponens* if we first obtain $E$. This, in turn, we can obtain from line 3 via disjunctive syllogism if we first obtain $\sim G$. Since $\sim G$ appears by itself on line 4, the first part of the thought process is now complete. The second part requires that we obtain $L \supset M$. This we can get from line 2 via *modus ponens* if we can get $F$, and we do have $F$ by itself on line 5. All of the steps leading to the conclusion have now been thought through, and the proof can be written out:

| | |
|---|---|
| 1. $E \supset (K \supset L)$ | |
| 2. $F \supset (L \supset M)$ | |
| 3. $G \vee E$ | |
| 4. $\sim G$ | |
| 5. $F$ | / $K \supset M$ |
| 6. $E$ | 3, 4, DS |
| 7. $K \supset L$ | 1, 6, MP |
| 8. $L \supset M$ | 2, 5, MP |
| 9. $K \supset M$ | 7, 8, HS |

The thought process behind these proofs illustrates an important point about the construction of proofs by natural deduction. As a rule, we should never write down a line in a proof unless we know why we are doing it and where it leads. Typically, good proofs are not produced haphazardly or by luck; rather, they are produced by organized logical thinking. Occasionally, of course, we may be baffled by an especially difficult proof, and random deductive steps noted on the side may be useful. But we should not commence the actual writing out of the proof until we have used logical thinking to discover the path leading to the conclusion.

## EXERCISE 7.1

**I.** Supply the required justification for the derived steps in the following proofs. No justification, of course, is required for the premises. The last premise is always the line adjacent to the required conclusion.

★(1)
1. $J \supset (K \supset L)$
2. $L \vee J$
3. $\sim L$     / $\sim K$
4. $J$     ______
5. $K \supset L$     ______
6. $\sim K$     ______

(2)
1. $\sim(S \equiv T) \supset (\sim P \supset Q)$
2. $(S \equiv T) \supset P$
3. $\sim P$     / $Q$
4. $\sim(S \equiv T)$     2,3 MT
5. $\sim P \supset Q$     1,4 MP
6. $Q$     3,5 MP

(3)
1. $\sim A \supset (B \supset \sim C)$
2. $\sim D \supset (\sim C \supset A)$
3. $D \vee \sim A$
4. $\sim D$     / $\sim B$
5. $\sim A$     3,4 DS
6. $B \supset \sim C$     1,5 MP
7. $\sim C \supset A$     2,4 MP
8. $B \supset A$     6,7 HS
9. $\sim B$     5,8 MT

★(4)
1. $\sim G \supset [G \vee (S \supset G)]$
2. $(S \vee L) \supset \sim G$
3. $S \vee L$     / $L$
4. $\sim G$     ______
5. $G \vee (S \supset G)$     ______
6. $S \supset G$     ______
7. $\sim S$     ______
8. $L$     ______

(5)
1. $H \supset [\sim E \supset (C \supset \sim D)]$
2. $\sim D \supset E$
3. $E \vee H$
4. $\sim E$     / $\sim C$
5. $H$     3,4 DS
6. $\sim E \supset (C \supset \sim D)$     1,5 MP
7. $C \supset \sim D$     4,6 MP
8. $C \supset E$     2,7 HS
9. $\sim C$     4,8 MT

**II.** Use the first four rules of inference to derive the conclusions of the following symbolized arguments:

★(1) 1. $(G \equiv J) \vee (B \supset P)$
2. $\sim(G \equiv J)$    $/ B \supset P$

(2) 1. $(K \cdot O) \supset (N \vee T)$
2. $K \cdot O$    $/ N \vee T$

(3) 1. $(M \vee P) \supset \sim K$
2. $D \supset (M \vee P)$    $/ D \supset \sim K$

★(4) 1. $\sim\sim(R \vee W)$
2. $S \supset \sim(R \vee W)$    $/ \sim S$

(5) 1. $\sim C \supset (A \supset C)$
2. $\sim C$    $/ \sim A$

(6) 1. $F \vee (D \supset T)$
2. $\sim F$
3. $D$    $/ T$

★(7) 1. $(K \cdot B) \vee (L \supset E)$
2. $\sim(K \cdot B)$
3. $\sim E$    $/ \sim L$

(8) 1. $P \supset (G \supset T)$
2. $Q \supset (T \supset E)$
3. $P$
4. $Q$    $/ G \supset E$

(9) 1. $\sim W \supset [\sim W \supset (X \supset W)]$
2. $\sim W$    $/ \sim X$

★(10) 1. $\sim S \supset D$
2. $\sim S \vee (\sim D \supset K)$
3. $\sim D$    $/ K$

(11) 1. $A \supset (E \supset \sim F)$
2. $H \vee (\sim F \supset M)$
3. $A$
4. $\sim H$    $/ E \supset M$

(12) 1. $N \supset (J \supset P)$
2. $(J \supset P) \supset (N \supset J)$
3. $N$    $/ P$

★(13) 1. $G \supset [\sim O \supset (G \supset D)]$
2. $O \vee G$
3. $\sim O$    $/ D$

(14) 1. $\sim M \vee (B \vee \sim T)$
2. $B \supset W$
3. $\sim\sim M$
4. $\sim W$    $/ \sim T$

(15) 1. $(L \equiv N) \supset C$
2. $(L \equiv N) \vee (P \supset \sim E)$
3. $\sim E \supset C$
4. $\sim C$    $/ \sim P$

★(16) 1. $\sim J \supset [\sim A \supset (D \supset A)]$
2. $J \vee \sim A$
3. $\sim J$    $/ \sim D$

(17) 1. $(B \supset \sim M) \supset (T \supset \sim S)$
2. $B \supset K$
3. $K \supset \sim M$
4. $\sim S \supset N$    $/ T \supset N$

(18) 1. $(R \supset F) \supset [(R \supset \sim G) \supset (S \supset Q)]$
2. $(Q \supset F) \supset (R \supset Q)$
3. $\sim G \supset F$
4. $Q \supset \sim G$    $/ S \supset F$

★(19) 1. $\sim A \supset [A \vee (T \supset R)]$
2. $\sim R \supset [R \vee (A \supset R)]$
3. $(T \vee D) \supset \sim R$
4. $T \vee D$    $/ D$

(20) 1. $\sim N \supset [(B \supset D) \supset (N \vee \sim E)]$
2. $(B \supset E) \supset \sim N$
3. $B \supset D$
4. $D \supset E$    $/ \sim D$

**III.** Translate the following arguments into symbolic form and use the first four rules of inference to derive the conclusion of each. The letters to be used for the atomic statements are given in parentheses after each exercise. Use these letters in the order in which they are listed.

★1. If the average child watches more than five hours of television per day, then either his power of imagination is improved or he becomes conditioned to expect constant excitement. The average child's power of imagination is not improved by watching television. Also, the average child does watch more than five hours of television per day. Therefore, the average child is conditioned to expect constant excitement. (*W*, *P*, *C*)

2. If a tenth planet exists, then its orbit is perpendicular to that of the other planets. Either a tenth planet is responsible for the death of the dinosaurs, or its orbit is not perpendicular to that of the other planets. A tenth planet is not responsible for the death of the dinosaurs. Therefore, a tenth planet does not exist. (*E*, *O*, *R*)

3. If imposing quotas on textile imports implies that jobs will not be lost, then the domestic textile industry will modernize only if the domestic textile industry is not destroyed. If quotas are imposed on textile imports, the domestic textile industry will modernize. The domestic textile industry will modernize only if jobs are not lost. Therefore, if quotas are imposed on textile imports, the domestic textile industry will not be destroyed. (*Q*, *J*, *M*, *D*)

★4. If teachers are allowed to conduct random drug searches on students only if teachers are acting *in loco parentis*, then if teachers are acting *in loco parentis*, then students have no Fourth Amendment protections. Either students have no Fourth Amendment protections or if teachers are allowed to conduct random drug searches on students, then teachers are acting *in loco parentis*. It is not the case that students have no Fourth Amendment protections. Therefore, teachers are not allowed to conduct random drug searches on students. (*R*, *L*, *F*)

5. Either funding for nuclear fusion will be cut or if sufficiently high temperatures are achieved in the laboratory, nuclear fusion will become a reality. Either the supply of hydrogen fuel is limited, or if nuclear fusion becomes a reality, the world's energy problems will be solved. Funding for nuclear fusion will not be cut. Furthermore, the supply of hydrogen fuel is not limited. Therefore, if sufficiently high temperatures are achieved in the laboratory, the world's energy problems will be solved. (*C*, *H*, *R*, *S*, *E*)

6. Either the continents are not subject to drift or if Antarctica was always located in the polar region, then it would contain no fossils of plants from a temperate climate. If the continents are not subject to drift, then Antarctica would contain no fossils of plants from a temperate climate. But it is not the case that Antarctica contains no fossils of plants from a temperate climate. Therefore, Antarctica was not always located in the polar region. (*D*, *L*, *F*)

★7. If terrorists take more hostages, then terrorist demands will be met if and only if the media give full coverage to terrorist acts. Either the media will voluntarily limit the flow of information or if the media will recognize they are being exploited by terrorists, they will voluntarily limit the flow of information. Either the media will recognize they are being exploited by terrorists or terrorists will take more hostages. The media will not voluntarily limit the flow of information. Therefore, terrorist demands will be met if and only if the media give full coverage to terrorist acts. (*H*, *D*, *A*, *V*, *R*)

8. Either the Soviets will delay developing a first-strike nuclear submarine, or they will be forced to launch their land-based missiles "on warning." If the Trident 2's ability to deliver nuclear warheads within 400 feet of targets implies that the Trident 2 is a first-strike weapon, then the Soviets will not delay developing a first-strike nuclear submarine. If Soviet land-based missiles are vulnerable to direct attack, then if the Trident 2 has the ability to deliver nuclear warheads within 400 feet of targets, then it can destroy Soviet land-based missiles. If the Trident 2 can destroy Soviet land-based missiles, then it is a first-strike weapon. Soviet land-based missiles are vulnerable to direct attack. Therefore, the Soviets will be forced to launch their land-based missiles "on warning." (*S, F, T, W, V, D*)

9. If strengthening the drug interdiction program implies that cocaine will become more readily available, then either the number of addicts is decreasing or the war on drugs is failing. If the drug interdiction program is strengthened, then smugglers will shift to more easily concealable drugs. If smugglers shift to more easily concealable drugs, then cocaine will become more readily available. Furthermore, the number of addicts is not decreasing. Therefore, the war on drugs is failing. (*D, C, N, W, S*)

★10. If the death penalty is not cruel and unusual punishment, then either it is cruel and unusual punishment or if society is justified in using it, then it will deter other criminals. If the death penalty is cruel and unusual punishment, then it is both cruel and unusual and its use degrades society as a whole. It is not the case that both the death penalty is cruel and unusual and its use degrades society as a whole. Furthermore, the death penalty will not deter other criminals. Therefore, society is not justified in using the death penalty. (*C, J, D, U*)

## 7.2 RULES OF IMPLICATION II

Four additional rules of inference are listed below. Constructive dilemma should be familiar from Chapter 6. The other three are new.*

5. Constructive dilemma (CD):

$$\begin{array}{l} (p \supset q) \cdot (r \supset s) \\ \underline{p \vee r} \\ q \vee s \end{array}$$

6. Simplification (Simp):

$$\begin{array}{l} \underline{p \cdot q} \\ p \end{array}$$

7. Conjunction (Conj):

$$\begin{array}{l} p \\ \underline{q} \\ p \cdot q \end{array}$$

8. Addition (Add):

$$\begin{array}{l} \underline{p} \\ p \vee q \end{array}$$

Like the previous four rules, these four are fairly easy to understand, but if there is any doubt about them their validity may be proven by means of a truth table.

*Some texts include a rule called "absorption" by which the statement form $p \supset (q \cdot p)$ is deduced from $p \supset q$. This rule is necessary only if conditional proof is not presented. This text opts in favor of conditional proof.